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Semicontinuity of set valued mappings and duality formulas of integral functionals

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測度の凸関数による汎関数の双対公式と集合値写像の半連続性

§1 DUALITY FORMULAS

Let X be a metric space, and let f be a real valued function defined on $X \times \mathbb{R}^d$. Suppose that for each $x \in X$, $f_x(p) = f(x,p)$ is convex and positively homogeneous in $p \in \mathbb{R}^d$. By K_x, we denote the subdifferential of f_x at 0;

$$\begin{aligned} K_x &= \partial f_x(0) \\ &= \{q \in \mathbb{R}^d \mid \ < q,p > \leq f_x(p), \quad p \in \mathbb{R}^d\} \end{aligned}$$

For every $x \in X$ the set K_x is convex in $\mathbb{R}^d$, and since $f_x(p)$ is finite for all $p \in \mathbb{R}^d$, K_x is compact. Let $\mu = (\mu_1, \cdots, \mu_n)$ be a $\mathbb{R}^d$ -valued finite Borel regular measure on X. The finite Borel measure $f(x,\mu)$ on X is defined by

$$\int_A f(x,\mu) = \int_A f(x, \overrightarrow{\mu(x)})d|\mu| \qquad \textit{for a Borel set } A \subset X$$

where $|\mu|$ is the total variation measure of μ and $\overrightarrow{\mu(x)} = \frac{d\mu}{d|\mu|}(x)$ is the Radon Nikodym derivative of μ with respect to $|\mu|$. The measure $f(x,\mu)$ is independent of the choice of a norm in $\mathbb{R}^d$.

THEOREM 1. *Suppose that f satisfies*

(1) f *is lower semicontinuous (l.s.c.) on* $X \times \mathbb{R}^d$,

(2) *for each* $x \in X$, $f_x(p) = f(x,p)$ *is convex, positively homogeneous in* p,

(3) $f(x,p) \le c|p| \qquad (x \in X, p \in \mathbb{R}^d)$ *with some constant* c.

Then for every bounded $|\mu|$*-measurable function* $\varphi \ge 0$ *on* X,

$$\text{(F,1)} \qquad \int_X f(x,\mu)\varphi = \sup\{\int_X < \overrightarrow{\mu(x)}, v(x) > \varphi(x) d|\mu|(x) \quad | \quad v \in C(X,\mathbb{R}^d),\ v(x) \in K_x \quad for\ all\ x \in X\ \}.$$

Next we consider the case when $f_x(\cdot)$ is only convex in $p \in \mathbb{R}^d$,and is not necessarily positively homogeneous. For definning the measure $f(x,\mu)$ in this case, we introduce the homogenization $F(x,p_0,p)$ of $f(x,p)$ defined by

$$F(x,p_0,p) = \begin{cases} f_\infty(x,p) & p_0 = 0 \\ f(x,\frac{p}{p_0})p_0 & p_0 > 0 \\ \infty & p_0 < 0 \end{cases}$$

where f_∞ is the recession function of f, i.e.,

$$f_\infty(x,p) = \lim_{t \downarrow 0} f(x,\frac{p}{t})t.$$

If f satifies $f(x,p) \le c(1+|p|) \qquad (x \in X, p \in \mathbb{R}^d)$ with some constant c, F is well-defined real valued function on $X \times C$ with $C = [0,\infty) \times \mathbb{R}^d$ and $F = \infty$ on $X \times (\mathbb{R}^{d+1} \setminus C)$. Moreover, F is convex and positively homogeneous in $(p_0,p) \in \mathbb{R}^{d+1}$. (See [8,§8])

Let α be a nonnegative finite Borel regular measure on X. We fix this mesure and now define the measure $f(x,\mu)$ by

$$f(x,\mu) = F(x,\alpha,\mu),$$

where F is the homogenization of f. Here (α,μ) is a $C = [0,\infty) \times \mathbb{R}^d$ valued Borel regular measure, and since F is positively homogeneous, $f(x,\mu)$ is a finite Borel regular measure.

It is easy to see that

$$\begin{aligned} f(x,\mu) &= F(x,\alpha,\mu) \\ &= F(x,1,\overrightarrow{h(x)})\alpha + F(x,0,\mu^s) \\ &= f(x,\overrightarrow{h(x)})\alpha + f_\infty(x,\overrightarrow{\mu^s}(x))|\mu^s| \end{aligned}$$

where $\overrightarrow{h(x)}\alpha$ is the absolutely continuous part of μ,and μ^s is the singular part with respect to α.

THEOREM 2. *Suppose that f satisfies*

(1) *for every $x_0 \in X$ and $\varepsilon > 0$, there is $\delta > 0$ such that $d(x,x_0) < \delta$ implies $f(x_0,p) - f(x,p) < \varepsilon(1+|p|)$,*

(2) *for each $x \in X$, $f_x(p)$ is convex in p,*

(3) *$f(x,p) \le c(1+|p|) \qquad (x \in X, p \in \mathbb{R}^d)$ with some constant c.*

Then for every bounded $|\mu|$-measurable function $\varphi \ge 0$ on X,

$$\text{(F,2)} \qquad \int_X f(x,\mu)\varphi = \sup\{\int_X <\overrightarrow{\mu(x)}, v(x)> \varphi(x)d|\mu|(x) - \int_X \varphi(x)f^*(x,v(x))d\alpha \quad | \quad v \in C(X,\mathbb{R}^d), f^*(x,v(x)) \in L^1(X,d\alpha)\ \}.$$

Similar results can be seen in [2], [3], [6]. In the proof of Rockafellar [6], it is assumed that K_x has an interior point and the assumption on the regularity of f in x is slightly stronger than ours. In [2], it is assumed that f is continuous on $X \times \mathbb{R}^d$. We have weakened these assumptions by some arguments of the continuous selection.

We consider the set valued mapping K which carries each $x \in X$ to the compact convex set $K_x \subset \mathbb{R}^d$. K is said to be lower semicontinuous (l.s.c.) if $x_n \longrightarrow x_0$ in X and $q_0 \in K_{x_0}$ implies the existence of a sequence $\{q_n\}$ such that $q_n \in K_{x_n}$ and $q_n \longrightarrow q_0$. K_x is said to be upper semicontinuous (u.s.c.) if for any sequence $\{x_n\}$ tends to x_0 and $\varepsilon > 0$, $K_{x_n} \subset K_{x_0} + \varepsilon B$ holds for sufficiently large n, where $K_{x_0} + \varepsilon B = \{q + q' \in \mathbb{R}^d \mid q \in K_{x_0},\ |q'| \le \varepsilon\}$. Furthermore, when K_x is both l.s.c. and u.s.c., K is said to be continuous. One can find some other definitions of this semicontinuity in [1], [5], and [6] for instance.

However, in our case, most of them are all equivalent because K_x is always compact. The importance of the lower semicontinuity is that this allows us to take continuous selection of K_x. For example, In [6], the lower semicontinuity of K_x and the continuous selection theorem ([5]) are applied to prove a type of duality formula. Also in [2], the conditions for the same formula are given in terms of the function $f(x,p)$. However, the relation between the conditions of these two theorems is unclear. In this note, we investigate the conditions of f under which K_x is lower semicontinuous. Moreover, we will consider the upper semicontinuity and derive some duality of these two notions.

§2 Semi continuity of K_x

LEMMA 3. *Let $f(x,p)$ be a function on $X \times \mathbb{R}^d$, and suppose that $f_x(p) = f(x,p)$ is convex and positively homogeneous in $p \in \mathbb{R}^d$. Put $K_x = \partial f_x(0)$, then the following conditions are equivalent.*

$(l,1)$ *f is l.s.c. on $X \times \mathbb{R}^d$.*

$(l,2)$ *For every $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x_0,x) < \delta$ implies*

$$f(x_o,p) - f(x,p) < \varepsilon|p|, \quad for\ all\ p \in \mathbb{R}^d.$$

$(l,3)$ *$K : x \longrightarrow K_x$ is l.s.c. on X.*

REMARK: When f is l.s.c. only in x, these conditions do not hold though f is convex (and hence continuous) in p. This fact is the only thing that the symmetry of Lemma 3 and Proposition 6 fails. The space $\mathbb{R}^d$ in this theorem can be replaced by any closed convex cone in $\mathbb{R}^d$, but not by any infinite dimensional space. Moreover, positively homogeneity of f is essential in this lemma even if K_x can be defined as the subdifferential of f.

PROOF: $(l,1) \Rightarrow (l,2)$

It suffices to show that $\{f(\cdot,p) \mid |p| = 1\}$ is equi l.s.c.. If not, there exists $x_0 \in X$, $\varepsilon > 0$, and sequences $\{x_n\} \subset X$ and $\{p_n\} \subset \mathbb{R}^d$, such that $x_n \longrightarrow x_0$, $|p_n| = 1$, and $f(x_0,p_n) - f(x_n,p_n) \geq \varepsilon\ for\ every\ n$. Since $\{p \in \mathbb{R}^d \mid |p| = 1\}$ is compact, we can assume that $p_n \longrightarrow p_0$ for some $|p_0|$. By the convexity of f in p, it is continuous in particular. Hence it follows by $(l,1)$ that

$$\begin{aligned} f(x_0,p_n)-f(x_n,p_n) &= f(x_0,p_n)-f(x_0,p_0)+f(x_0,p_0)-f(x_n,p_n) \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon, \end{aligned}$$

for sufficiently large n and this contradicts the assumption.

$(l,2) \Rightarrow (l,3)$

Suppose that K is not l.s.c. at $x_0 \in X$. Then there exist a sequence $\{x_n\}$ with $x_n \longrightarrow x_0$, $q_0 \in K_{x_0}$ and $\varepsilon > 0$ such that

$$Kx_n \cap \varepsilon B(q_0) = \phi, \tag{1}$$

for every n, where $\varepsilon B(q_0) = \{q \in \mathbb{R}^d \mid d(q,q_0) \leq \varepsilon\}$. By the condition $(l,2)$, we have for sufficiently large n,

$$f(x_0,p)-f(x_n,p) < \varepsilon \qquad for\ all\ p \in \mathbb{R}^d\ with\ |p|=1. \tag{2}$$

We fix such n, and by the separation theorem and (1), there exists $p_0 \in \mathbb{R}^d$ with $|p_0|=1$, such that

$$\sup_{q\in K_{x_n}} < q,p_0 > \leq \inf_{q\in\varepsilon B(q_0)} < q,p_0 > . \tag{3}$$

Now we take the supporting point $\bar{q}$ of $\varepsilon B(q_0)$ with respect to p_0, that is, $\bar{q} \in \varepsilon B(q_0)$ and $\inf_{q\in\varepsilon B(q_0)} < q,p_0 > = < \bar{q},p_0 >$. Then,

$$\begin{aligned} \inf_{q\in\varepsilon B(q_0)} < q,p_0 > &= < q_0,p_0 > - < q_0-\bar{q},p_0 > \\ &= < q_0,p_0 > -\varepsilon \\ &\leq \sup_{q\in K_{x_0}} < q,p_0 > -\varepsilon \\ &= f(x_0,p_0)-\varepsilon. \end{aligned}$$

By (3), we obtain

$$f(x_n,p_0) \leq f(x_0,p_0)-\varepsilon.$$

Since p in (2) is arbitrary, this is a contradiction.

$(l,3) \Rightarrow (l,1)$

Suppose that $x_n \longrightarrow x_0$ in X and $p_n \longrightarrow p_0$ in $\mathbb{R}^d$. For every $\varepsilon > 0$, we take $q_0 \in K_{x_0}$ such that

$$
\begin{aligned}
< q_0, p_0 > &\geq \sup_{q \in K_{x_0}} < q, p_0 > -\varepsilon \\
&= f(x_0, p_0) - \varepsilon.
\end{aligned}
$$

By $(l,3)$, there exists a sequence $\{q_n\}$ such that each q_n belongs to K_{x_n} and $q_n \longrightarrow q_0$. Since $< q_n, p_n > \leq \sup_{q \in K_{x_n}} < q, p_n > = f(x_n, p_n)$, we have

$$
\begin{aligned}
f(x_0, p_0) - f(x_n, p_n) &\leq < q_0, p_0 > + \varepsilon - < q_n, p_n > \\
&< 2\varepsilon
\end{aligned}
$$

for sufficiently large n. This implies that f is l.s.c. on $X \times \mathbb{R}^d$. ∎

COROLLARY 4. *Suppose that f satisfies one of three conditions in Theorem 3. Then for every $x_0 \in X$ and $p_0 \in \mathbb{R}^d$, there exists a continuous function L on $X \times \mathbb{R}^d$ satisfying*

(1) for every $x \in X$, $L(x,p)$ is linear in $p \in \mathbb{R}^d$,

(2) $L(x,p) \leq f(x,p) \quad$ for all $x \in X$ and $p \in \mathbb{R}^d$,

(3) $L(x_0, p_0) = f(x_0, p_0)$.

PROOF: First we note that L is continuous on $X \times \mathbb{R}^d$ if it satifies (1) and is continuous with respect to each variable. By the separation theorem or Hahn Banach theorem, there exists $q_0 \in \mathbb{R}^d$ such that $< q_0, p > \leq f(x_0, p)$ for all $p \in \mathbb{R}^d$, and $< q_0, p_0 > = f(x_0, p_0)$. Take a set valued mapping K' defined by

$$
K'_x = \begin{cases} K_x & x \neq x_0 \\ \{q_0\} & x = x_0. \end{cases}
$$

Since $q_0 \in K_{x_0}$, it is easy to see that K' is l.s.c., and hence we can take a continuous selection $q(x)$ of K'_x. Thus the function L defined by $L(x,p) = < q(x), p > \quad (x \in X, p \in \mathbb{R}^d)$ is what we want. ∎

By an analogy, one can also prove the following.

Corollary 5. *Suppose that f satisfies one of the three conditions in Theorem 3. Let E be a closed subset of X, and let L be a continuous function on $E \times \mathbb{R}^d$ satisfying*

(1) for every $x \in E$, $L(x,p)$ is linear in $p \in \mathbb{R}^d$,

(2) $L(x,p) \leq f(x,p)$ for all $x \in E$ and $p \in \mathbb{R}^d$.

Then L can be continuously extended to $X \times \mathbb{R}^d$ such that (1) *and* (2) *hold replacing E by X.*

Next we consider the upper semicontinuity of K_x. We note that the following proposition and Lemma 3 have some symmetricity but it is not perfect.

Proposition 6. *Under the hypotheses in Lemma 3, the following conditions are equivalent.*

$(u,0)$ *For every $p \in \mathbb{R}^d$, $f(x,p)$ is u.s.c. in $x \in X$.*

$(u,1)$ *f is u.s.c. on $X \times \mathbb{R}^d$.*

$(u,2)$ *For every $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x_0,x) < \delta$ implies*

$$f(x,p) - f(x_0,p) < \varepsilon|p|, \quad \textit{for all } p \in \mathbb{R}^d.$$

$(u,3)$ *$K : x \longrightarrow K_x$ is u.s.c. on X.*

Remark: A set valued mapping K is said to be closed if for any sequence $\{x_n\}$ with $x_n \longrightarrow x_0$, and $\{q_n\}$ with $q_n \in K_{x_n}$, $q_n \longrightarrow q_0$ for some $q_0 \in \mathbb{R}^d$ implies $q_0 \in K_{x_0}$. This is also a notion of upper semicontinuity of set valued mappings. Since K_x is compact in our case, the upper semicontinuity of K implies the closedness. However, the converse is not true in general. The equivalence of $(u,0)$ and $(u,1)$ is still valid when f is only convex and not positively homogeneous in p.

Proof: $(u,0) \Rightarrow (u,1)$

Suppose that $x_n \longrightarrow x_0$ in X and $p_n \longrightarrow p_0$ in $\mathbb{R}^d$. Since f is continuous in p, there exists $\bar{p}_1, \cdots, \bar{p}_{d+1} \in \mathbb{R}^d$ such that

$$f(x_0,\bar{p}_i) \leq f(x_0,p_0) + \frac{\varepsilon}{2} \qquad (i = 1, \cdots, d+1)$$

and the convex hull $co\{\bar{p}_1, \cdots, \bar{p}_{d+1}\}$ forms a neighborhood of p_0. Moreover by the condition $(u, 0)$,

$$f(x_n, \bar{p}_i) \leq f(x_0, \bar{p}_i) + \frac{\varepsilon}{2} \qquad (i = 1, \cdots, d+1)$$

holds for sufficiently large n. Since $p_n \in co\{\bar{p}_1, \cdots, \bar{p}_{d+1}\}$ for sufficiently large n, we obtain by the convexity of $f(x, \cdot)$ that

$$\begin{aligned} f(x_n, p_n) &\leq \max_{1 \leq i \leq d+1} f(x_n, \bar{p}_i) \\ &\leq \max_{1 \leq i \leq d+1} f(x_0, \bar{p}_i) + \frac{\varepsilon}{2} \\ &\leq f(x_0, p_0) + \varepsilon. \end{aligned}$$

This proves that $(u, 1)$ holds.

$(u, 1) \Rightarrow (u, 2)$

we can prove this by the same way as in $(l, 1) \Rightarrow (l, 2)$ in Lemma 3.

$(u, 2) \Rightarrow (u, 3)$

Take $x_0 \in X$ and $\varepsilon > 0$ arbitrarily, and Suppose that $x_n \longrightarrow x_0$ *in* X. By $(u, 2)$,

$$f(x_n, p) - f(x_0, p) \leq \varepsilon |p| \qquad (p \in \mathbb{R}^d),$$

for sufficiently large n. Then $q \in K_{x_n}$ implies that

$$f(x_0, p) - < q, p > \geq f(x_0, p) - f(x_n, p) > -\varepsilon |p| \; for \; all \; p \in \mathbb{R}^d.$$

By the separation theorem, there exists $q_0 \in \mathbb{R}^d$ such that

$$-\varepsilon |p| \leq < q_0, p > \leq f(x_0, p) - < q, p > \qquad (p \in \mathbb{R}^d).$$

This inequality implies that $|q_0| \leq \varepsilon$, and $q + q_0 \in K_{x_0}$. Hence we have $q \in K_{x_0} + \varepsilon B$ and this proves $(u, 3)$.

$(u, 3) \Rightarrow (u, 1)$

For the reason stated in the remark of this theorem, we can assume that K is closed. Suppose that $(u, 1)$ does not hold, then there exist sequences $\{x_n\}$ with $x_n \longrightarrow x_0$ for some x_0 *in* X, and $\{p_n\}$ with $p_n \longrightarrow p_0$ for some p_0 *in* $\mathbb{R}^d$, and $\varepsilon > 0$ such that $f(x_n, p_n) >$

$f(x_0, p_0) + \varepsilon$ for every n. Since $f(x_n, p_n) = \sup_{q \in K_{x_n}} < q, p_n >$, we can choose a sequence $\{q_n\} \subset \mathbb{R}^d$ such that $q_n \in K_{x_n}$ and

$$| f(x_n, p_n) - < q_n, p_n > | \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

By the definition of upper semicontinuity, K_{x_n} is uniformly bounded. Therefore the sequence $\{q_n\}$ is bounded, and we can take a convergent subsequence $\{q_m\}$ of $\{q_n\}$ with $q_m \longrightarrow q_0$ for some $q_0 \in \mathbb{R}^d$. Hence it follows that

$$< q_0, p_0 > \geq f(x_0, p_0) + \varepsilon.$$

On the oter hand, by the closedness of K, q_0 has to be an element of K_{x_0}, and this is a contradiction. ∎

Combining Lemma 3 and Proposition 6, we also obtain the following theorem. To see the equivalence between $(c, 0)$ and $(c, 1)$, refer to Theorem 1.1 in [3].

PROPOSITION 7. *Under the hypothses in Lemma 3, the following conditions are equivalent.*

$(c, 0)$ *For every $p \in \mathbb{R}^d$, $f(x, p)$ is continuous in $x \in X$.*

$(c, 1)$ *f is continuous on $X \times \mathbb{R}^d$.*

$(c, 2)$ *For every $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x_0, x) < \delta$ implies*

$$| f(x, p) - f(x_0, p)| < \varepsilon |p|, \quad \textit{for all } p \in \mathbb{R}^d.$$

$(c, 3)$ *$K : x \longrightarrow K_x$ is continuous on X.*

§3 PROOF OF THE DUALITY FORMULA

For a subset $U \subset \mathbb{R}^d$, we denote the inverse image of a set valued mapping K by

$$K^{-1}(U) = \{x \in X | K_x \cap U \neq \phi\}.$$

K is l.s.c. if and only if $K^{-1}(U)$ is open for every Open set $U \subset \mathbb{R}^d$. Moreover, we say K is $|\mu|$-measurable if $K^{-1}(U)$ is $|\mu|$- measurable for every open set $U \subset \mathbb{R}^d$. For the detail of the continuous selection theorem and the measurable selection theorem, we refer to[1],[5],and [7].

PROOF OF THEOREM 1: Note that '$\geq$' part of the formulas are almost trivial and it suffices to prove the converse inequality. First we show a weaker version of the formula $(F, 1)$ while the supremum is taken over $|\mu|$-measurable function $w : X \longrightarrow \mathbb{R}^d$ with $w(x) \in K_x$. For arbitrary $\varepsilon > 0$, and $x \in X$, put

$$\Gamma(x) = \{p \in \mathbb{R}^d | < \overrightarrow{\mu(x)}, p > \geq f_x(\overrightarrow{\mu(x)}) - \varepsilon\},$$
$$\Gamma_0(x) = \{p \in K_x | < \overrightarrow{\mu(x)}, p > \geq f_x(\overrightarrow{\mu(x)}) - \varepsilon\}.$$

Since $f_x(\overrightarrow{\mu(x)}) = \sup_{p \in K_x} < \overrightarrow{\mu(x)}, p >$, $\Gamma(x)$ and $\Gamma_0(x)$ are nonempty closed convex sets in $\mathbb{R}^d$, and $\Gamma(x) = \Gamma_0(x) \cap K_x$. By the condition (1) and Lemma 3, K is l.s.c. as a set valued mapping, and also measurable in particular. Hence by [7, Theorm 1M], Γ is a $|\mu|$ -measurable set valued mapping provided that so is Γ_0. Let U be an open set in $\mathbb{R}^d$. Since $\Gamma_0(x)$ is an affine half space, $\Gamma_0(x) \cap U \neq \phi$ if and only if $\Gamma_0(x) \cap D \neq \phi$ where D is an arbitrary countable dense subset of U. Hence we have

$$\begin{aligned}\Gamma_0^{-1}(U) &= \Gamma_0^{-1}(D)\\ &= \bigcup_{p \in D} A_p\end{aligned}$$

where $A_p = \{x \in X | < \overrightarrow{\mu(x)}, p > \geq f_x(\overrightarrow{\mu(x)}) - \varepsilon\}$. We note that $f_x(\overrightarrow{\mu(x)})$ is $|\mu|$ -measurable because of the lower semicontinuity of f. Thus $\Gamma_0^{-1}(U)$ is $|\mu|$ -measurable, and by the measurable selection theorem we can take a measurable function w on Xsuch that $w(x) \in \Gamma(x)$. In other words

$$\begin{aligned}\int_X < \overrightarrow{\mu(x)}, w(x) > \varphi(x) d|\mu| &\geq \int_X (f_x(\overrightarrow{\mu(x)}) - \varepsilon)\varphi(x) d|\mu|\\ &= \int_X f(x, \mu)\varphi - \varepsilon \int_X \varphi d|\mu| \qquad (4)\end{aligned}$$

Since $|\mu|$ is finite measura and φ is bounded, this yields the duality formula of weaker version.

We next construct a desired continuous function $v : X \longrightarrow \mathbb{R}^d$ from w which has been obtained above. By Lusin's theorem, for arbitrary $\delta > 0$ there exists a closed set $Y \subset X$ such that $|\mu|(Y^c) < \delta$ and w is continuous on Y. We define a set valued mapping K' by

$$K'_x = \begin{cases} \{w(x)\} & x \in Y \\ K_x & x \notin Y \end{cases}$$

for $x \in X$. We see by [1, Corollary 9.1.3] (the closedness of K is missing in the condition of this corollary) that K' is also l.s.c. and have a continuous selection. In other words, there exists a contionuous function $v : X \longrightarrow \mathbb{R}^d$ such that $v(x) \in K_x$ on X and $v(x) = w(x)$ on Y. Hence we have

$$\begin{aligned}\int_X < \overrightarrow{\mu(x)}, w(x) > \varphi d|\mu| &= \int_X < \overrightarrow{\mu(x)}, v(x) > \varphi d|\mu| + \int_{Y^c} < \overrightarrow{\mu(x)}, w(x) > \varphi d|\mu| \\ &- \int_{Y^c} < \overrightarrow{\mu(x)}, v(x) > \varphi d|\mu| \\ &\leq \int_X < \overrightarrow{\mu(x)}, w(x) > \varphi d|\mu| + \int_{Y^c} f(x, \overrightarrow{\mu(x)})\varphi d|\mu| \\ &+ ||v|| \int_{Y^c} \varphi d|\mu|.\end{aligned}$$

Since $f(x,p) \leq c$ for $x \in X$ and $|p| = 1$, we thus obtain from (4) that

$$\begin{aligned}\int_X f(x,\mu)\varphi \leq \int_X &< \overrightarrow{\mu(x)}, v(x) > \varphi d|\mu| + (c + ||v||)||\varphi|||\mu|(Y^c) \\ &+ \varepsilon||\varphi|||\mu|(X).\end{aligned}$$

We note that $v(x) \in K_x$ implies $||v|| = \sup_{x\in X} |v(x)| \leq c$, which is independent of δ and ε. Since ε and δ are arbitrary, this yields the desired formula (F,1). ∎

The formula (F,1) is still valid in the case when the effective domain of $f_x(\cdot)$ is a closed convex cone $C \subset \mathbb{R}^d$. The proof can be done by a similar way except some standard arguments. Moreover, the formula (F,1) of this case is used for the proof of Theorem 2. Indeed, under the conditions in Theprem 2, the homogenization $F(x, p_0, p)$ satisfies the conditions in Theorem 1 by replacing $\mathbb{R}^d$ by the cone $C = [0, \infty) \times \mathbb{R}^d$, and we can apply Theorem 1 for F. To end this note, we show this fact in the following proposition.

Proposition 8. *If f satisfies (1),(2),(3) in Theorem 2, then the homogenization F satisfies (1),(2),(3) in Theorem 1 by replacing $\mathbb{R}^d$ by $C = [0,\infty) \times \mathbb{R}^d$.*

Proof: It is stated in §1 that F satisfies (2). Moreover,

$$\begin{aligned} F(x,0,p) &= \lim_{t\downarrow 0} f(x,\frac{p}{t})t \\ &\le \lim_{t\downarrow 0} c(1+|\frac{p}{t}|)t \\ &= c|p|, \\ F(x,p_0,p) &= f(x,\frac{p}{p_0})p_0 \\ &\le c(1+|\frac{p}{p_0}|)p_0 \\ &= c(|p_0|+|p|) \quad (p_0 \ne 0), \end{aligned}$$

and this proves (3). Hence it remains to prove (1). It is easy to see that F is l.s.c. in $(p_0,p) \in C = [0,\infty) \times \mathbb{R}^d$. Hence it follows from (1) in Theorem 2 that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|(p_0,p)-(q_0,q)| < \delta, d(x_0,x) < \delta, q_0 \ne 0$ implies

$$\begin{aligned} F(x_0,p_0,p) - F(x,q_0,q) &= F(x_0,p_0,p) - F(x_0,q_0,q) + F(x_0,q_0,q) - F(x,q_0,q) \\ &< \varepsilon + (f(x_0,\frac{q}{q_0}) - f(x,\frac{q}{q_0}))q_0 \\ &< \varepsilon + \varepsilon(1+|\frac{q}{q_0}|)q_0 \\ &= \varepsilon + \varepsilon(|q_0|+|q|). \end{aligned}$$

It is similar in the case of $q_0 = 0$. So F is l.s.c. on $X \times C$ and the proof is complete. ∎

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